e content for students of patliputra university

B. Sc. (Honrs) Part 1 paper 1

Subject:Mathematics

Title/Heading of topic: Sum and product of matrices

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2 sum and product of matrices

In this section, we will do some algebra of matrices. That means, we will add, subtract, multiply matrices. Matrices will usually be denoted by upper case letters, $A, B, C, \ldots$. Such a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

is also denoted by $[a_{ij}]$.

Definition 2.1.1 *Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if both A and B have same size $m \times n$ and the entries*

$$a_{ij} = b_{ij} \quad \text{for all} \quad 1 \leq i \leq m \quad \text{and} \quad 1 \leq j \leq n.$$

Definition 2.1.2 Following are some standard terminologies:

1. A matrix with only one column is called a **column matrix** or **column vector**. For example,

$$\mathbf{a} = \begin{bmatrix} 13 \\ 19 \\ 23 \end{bmatrix}$$

is a column matrix.

2. A matrix with only one row is called a **row matrix** or **row vector**. For example

$$\mathbf{b} = \begin{bmatrix} 4 & 11 & 13 & 19 & 23 \end{bmatrix}$$

is a row matrix.

2.1.1 Matrix Addition

Definition 2.1.3 We difine addition of two matrices of **same size.** Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of same size $m \times n$. Then the **sum** $A + B$ is defined to be the matrix of size $m \times n$ given by

$$A + B = [a_{ij} + b_{ij}].$$

1. For example, with

$$A = \begin{bmatrix} 4 & -5 \\ 3 & -8 \\ 10 & 14 \end{bmatrix}, \ B = \begin{bmatrix} 9 & 15 \\ -5 & 18 \\ 11 & 1 \end{bmatrix} \ \textit{we have } A{+}B = \begin{bmatrix} 13 & 10 \\ -2 & 10 \\ 21 & 15 \end{bmatrix}$$

2. Also, for example, with

$$C = \begin{bmatrix} -3.5 & -2 \\ 3 & -2.2 \end{bmatrix}, \ D = \begin{bmatrix} 0.5 & 2.7 \\ -3 & -5 \end{bmatrix} \ \textit{we have } C{+}D = \begin{bmatrix} -3 & 0.7 \\ 0 & -7.2 \end{bmatrix}.$$

3. While, the sum $A + C$ is not defined because A and C do not have same size.

2.1.2 Scalar Multiplication

Recall, in some contexts, real numbers are referred to as **scalars** (*in contrast with vectors.*) We define, multiplication of a matrix A by a scalar c.

Definition 2.1.4 Let $A = [a_{ij}]$ be an $m \times n$ matrix and c be a scalar. We define **Scalar multiple** cA of A by c as the matrix of same size given by

$$cA = [ca_{ij}].$$

1. For a matrix A, the negative of $-A$ denotes $(-1)A$. Also $A - B := A + (-1)B$.

2. Let $c = 11$ and

$$A = \begin{bmatrix} 4 & -5 \\ 3 & -8 \\ 10 & 14 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & 15 \\ -5 & 18 \\ 11 & 1 \end{bmatrix}$$

Then, with $c = 11$ we have

$$cA = 11 \begin{bmatrix} 4 & -5 \\ 3 & -8 \\ 10 & 14 \end{bmatrix} = \begin{bmatrix} 44 & -55 \\ 33 & -88 \\ 110 & 154 \end{bmatrix}.$$

Likewise, $A - B = A + (-1)B =$

$$\begin{bmatrix} 4 & -5 \\ 3 & -8 \\ 10 & 14 \end{bmatrix} + (-1) \begin{bmatrix} 9 & 15 \\ -5 & 18 \\ 11 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 3 & -8 \\ 10 & 14 \end{bmatrix} + \begin{bmatrix} -9 & -15 \\ 5 & -18 \\ -11 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-9 & -5-15 \\ 3+5 & -8-18 \\ 10-11 & 14-1 \end{bmatrix} = \begin{bmatrix} -5 & -20 \\ 8 & -26 \\ -1 & 13 \end{bmatrix}$$

2.1.3 Matrix Multiplication

Definition 2.1.5 Suppose $A = [a_{ij}]$ is a matrix of size $m \times n$ and $B = [b_{ij}]$ is a matrix of size $n \times p$. Then the product $AB = [c_{ij}]$ is a matrix size $m \times p$ where

$$c_{ij} := a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

1. Note that number of columns (n) of A must be equal to number of rows (n) of B, for the product AB to be defined.

2. Note the number of rows of AB is equal to is same as that (m) of A and number of columns of AB is equal to is same as that (n) of B.

Exercise 2.1.6 Let

$$A = \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

Compute AB abd BA. We have

$$AB = \begin{bmatrix} 1*1+(-1)*2+7*1 & 1*1+(-1)*1+7*(-3) & 1*2+(-1)*1+7*2 \\ 2*1+(-1)*2+8*1 & 2*1+(-1)*1+8*(-3) & 2*2+(-1)*1+8*2 \\ 3*1+1*2+(-1)*1 & 3*1+1*1+(-1)*(-3) & 3*2+1*1+(-1)*2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -21 & 15 \\ 8 & -23 & 19 \\ 4 & 1 & 5 \end{bmatrix}$$

Now, we compute BA. We have

$$BA = \begin{bmatrix} 1*1+1*2+2*3 & 1*(-1)+1*(-1)+2*1 & 1*7+1*8+2*(- \\ 2*1+1*2+1*3 & 2*(-1)+1*(-1)+1*1 & 2*7+1*8+1*(- \\ 1*1+(-3)*2+2*3 & 1*(-1)+(-3)*(-1)+2*1 & 1*7+(-3)*8+2*(\end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 & 13 \\ 7 & -2 & 21 \\ 1 & 4 & -19 \end{bmatrix}.$$

Also note that $AB \neq BA$.

Exercise 2.1.7 Let

$$A = \begin{bmatrix} 0 & -1 & 3 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

Compute AB and BA, if definded.

Solution: Note BA is not defined, because size of B is 3×1 and size of A is 3×3. But AB is defined and is a 3×1 matrix (or a column matrix). We have

$$AB = \begin{bmatrix} 0*2+(-1)*(-3)+3*1 \\ 4*2+0*(-3)+2*1 \\ 8*2+(-1)*(-3)+7*1 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 26 \end{bmatrix}.$$

Exercise 2.1.8 Let

$$A = \begin{bmatrix} 1 & 0 & 3 & -2 & 4 \\ 6 & 13 & 8 & -17 & 20 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix}$$

algebra of matrices

Theorem 2.2.1 (Properties) Let A, B, C be three $m \times n$ matrices and c, d be scalars. Then

1.	$A + B = B + A$	*Commutativity of matrix addition*
2.	$(A + B) + C = A + (B + C)$	*Associativity of matrix addition*
3.	$(cd)A = c(dA)$	*Associativity of scalar multiplication*
4.	$1A = A$	*identity of scalar multiplication*
5.	$c(A + B) = cA + cB$	*Distributivity of scalar multiplication*
6.	$(c + d)A = cA + dA$	*Distributivity of scalar multiplication*

Proof. One needs to prove all these statements using definitions of addition and scalar multiplication. To prove the commutativity of matrix addition (1), Let $A = [a_{ij}], B = [b_{ij}]$. Both A and B have same size $m \times n$, so $A + B, B + A$ are defined. From definition

$$A+B = [a_{ij}]+[b_{ij}] = [a_{ij}+b_{ij}] \quad \textit{and} \quad B+A = [b_{ij}]+[a_{ij}] = [b_{ij}+a_{ij}].$$

From commutative property of addition of real numbers, we have $a_{ij} + b_{ij} = b_{ij}+a_{ij}$. Therefore, from definition of equality of matrices, $A+B = B + A$. So, (1) is proved. Other properties are proved similarly. ∎

Remark. For matrices A, B, C as in the theorem, by the expression $A + B + C$ we mean $(A + B) + C$ or $A + (B + C)$. It is well defined, because $(A + B) + C = A + (B + C)$ by associatieve property(2) of matrix addition.

Theorem 2.2.2 Let O_{mn} denote the $m \times n$ matrix whose entries are all zero. Let A be a $m \times n$ matrix. Then

1. Then $A + O_{mn} = A$.

2. We have $A + (-A) = O_{mn}$

3. If $cA = O_{mn}$, then either $c = 0$ or $A = 0$.

We say, on the set of all $m \times n$ matrices, O_{mn} is an additive identity (property (1)), and $(-A)$ is the additive inverse of A (property (2))

2.2.1 Properties of matrix multiplication

Theorem 2.2.3 (Mult-Properties) Let A, B, C be three matrices (of varying sizes) so that all the products below are defined and c be a scalars. Then

1. $AB \neq BA$ — **Failure of** *Commutativity of multiplication*
2. $(AB)C = A(BC)$ — *Associativity of multiplication*
3. $(A+B)C = AC + BC$ — *Left – Distributivity of multiplication*
4. $A(B+C) = AB + AC$ — *Right – Distributivity of multiplication*
5. $c(AB) = (cA)B$ — *an Associativity*

In (1), by $AB \neq BA$, we mean AB is not always equal to BA.

Proof. We will only prove (4). In this case, let A be a matrix of size $m \times n$ and then B, C would have to be of same size $n \times p$. Write

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix},$$

and

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1p} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2p} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3p} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{np} \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1p} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2p} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3p} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{np} \end{bmatrix}.$$

Therefore, $A(B+C) =$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} & b_{13}+c_{13} & \cdots & b_{1p}+c_{1p} \\ b_{21}+c_{21} & b_{22}+c_{22} & b_{23}+c_{23} & \cdots & b_{2p}+c_{2p} \\ b_{31}+c_{31} & b_{32}+c_{32} & b_{33}+c_{33} & \cdots & b_{3p}+c_{3p} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1}+c_{n1} & b_{n2}+c_{n2} & b_{n3}+c_{n3} & \cdots & b_{np}+c_{np} \end{bmatrix}.$$

which is

$$= \begin{bmatrix} \sum_{k=1}^{n} a_{1k}(b_{k1}+c_{k1}) & \sum_{k=1}^{n} a_{1k}(b_{k2}+c_{k2}) & \cdots & \sum_{k=1}^{n} a_{1k}(b_{kp}+c_{kp}) \\ \sum_{k=1}^{n} a_{2k}(b_{k1}+c_{k1}) & \sum_{k=1}^{n} a_{2k}(b_{k2}+c_{k2}) & \cdots & \sum_{k=1}^{n} a_{2k}(b_{kp}+c_{kp}) \\ \sum_{k=1}^{n} a_{3k}(b_{k1}+c_{k1}) & \sum_{k=1}^{n} a_{3k}(b_{k2}+c_{k2}) & \cdots & \sum_{k=1}^{n} a_{3k}(b_{kp}+c_{kp}) \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{k=1}^{n} a_{mk}(b_{k1}+c_{k1}) & \sum_{k=1}^{n} a_{mk}(b_{k2}+c_{k2}) & \cdots & \sum_{k=1}^{n} a_{mk}(b_{kp}+c_{kp}) \end{bmatrix}$$

which is

$$
= \begin{array}{c}
\begin{bmatrix}
\sum_{k=1}^n a_{1k}b_{k1} & \sum_{k=1}^n a_{1k}b_{k2} & \cdots & \sum_{k=1}^n a_{1k}b_{kp} \\
\sum_{k=1}^n a_{2k}b_{k1} & \sum_{k=1}^n a_{2k}b_{k2} & \cdots & \sum_{k=1}^n a_{2k}b_{kp} \\
\sum_{k=1}^n a_{3k}b_{k1} & \sum_{k=1}^n a_{3k}b_{k2} & \cdots & \sum_{k=1}^n a_{3k}b_{kp} \\
\cdots & \cdots & \cdots & \cdots \\
\sum_{k=1}^n a_{mk}b_{k1} & \sum_{k=1}^n a_{mk}b_{k2} & \cdots & \sum_{k=1}^n a_{mk}b_{kp}
\end{bmatrix} \\
+ \\
\begin{bmatrix}
\sum_{k=1}^n a_{1k}c_{k1} & \sum_{k=1}^n a_{1k}c_{k2} & \cdots & \sum_{k=1}^n a_{1k}c_{kp} \\
\sum_{k=1}^n a_{2k}c_{k1} & \sum_{k=1}^n a_{2k}c_{k2} & \cdots & \sum_{k=1}^n a_{2k}c_{kp} \\
\sum_{k=1}^n a_{3k}c_{k1} & \sum_{k=1}^n a_{3k}c_{k2} & \cdots & \sum_{k=1}^n a_{3k}c_{kp} \\
\cdots & \cdots & \cdots & \cdots \\
\sum_{k=1}^n a_{mk}c_{k1}) & \sum_{k=1}^n a_{mk}c_{k2} & \cdots & \sum_{k=1}^n a_{mk}c_{kp}
\end{bmatrix}
\end{array} = AB + AC
$$

So, $A(B+C) = AB + AC$ and the proof of complete. ■

Alternate way to write the same proof: Let A be a matrix of size $m \times n$ and then B, C would have to be of same size $n \times p$. Write

$$A = [a_{ik}], \quad B = [b_{kj}], \quad C = [c_{kj}].$$

Then $B + C = [b_{kj} + c_{kj}]$. So,

$$A(B+C) = [a_{ik}][b_{kj} + c_{kj}] = [\alpha_{ij}] \ (say).$$

Then the $(ij)^{th}$ entry α_{ij} of $A(B+C)$ is given by

$$\alpha_{ij} = \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$$

But

$$AB = \left[\sum_{k=1}^{n} a_{ik}b_{kj}\right], \qquad and \qquad AC = \left[\sum_{k=1}^{n} a_{ik}c_{kj}\right]$$

So, the $(ij)^{th}$ entry of $AB + AC$ is also equal to α_{ij}. Therefore $A(B+C) = AB + AC$ and the proof is complete. ■